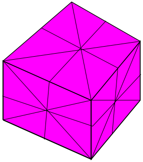
Subdivisions of Shellable Complexes



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arXiv:2003.07328

What is an h-polynomial?

Let Δ be a $(d-1)$-dimensional simplicial complex:

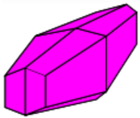
- f-poynomial — $f(\Delta; x) = f_{-1} + f_0 x + \cdots + f_{d-1} x^d$ where $f_{-1} = 1$ and f_i is the number of i-dimensonal faces of Δ.
- h-polynomial — $h(\Delta; x) = (1-x)^d f\left(\frac{x}{1-x}\right)$

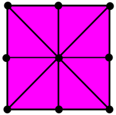
Classification questions that we may ask about h-polynomials:

- unimodality: $h_0 \leq h_1 \leq \cdots \leq h_k \geq \cdots \geq h_d$
- real-rooted-ness: stronger than unimodality

Motivation

Theorem (Brenti, Welker 2008) If $h(\Delta; x)$ has positive coefficients,the barycentric subdivision of Δ has a real-rooted h-polynomial.





Question: What other polytopal complexes have a barycentric subdivision with a real-rooted h-polynomial? One place to start:

- barycentric subdivision of boundary complexes of cubical polytopes (Brenti, Welker)
- cubical polytopes — All faces are combinatorially equivalent to cubes!

Sketch of technique

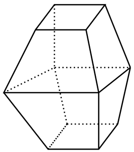
- Use the idea of shelling to decompose our subdivision into disjoint pieces.

- Find all possible h- polynomials of these bite-sized pieces

- Use the idea of interlacing polynomials to show that the $h-$ polynomial of the whole subdivision is real-rooted.

What is a shelling?

Let $\mathcal{C}$ be a pure d-dim polytopal complex.



Shelling— A linear ordering $(F_1, \ldots, F_s)$ on the facets of $\mathcal{C}$ such that:

1. Boundary complex of F_1 is shellable.
2. The intersection of F_j with the prev. facets is nonempty and is the beginning segment of a shelling of the boundary complex of F_j.

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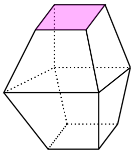

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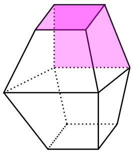

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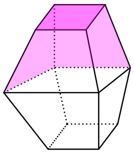

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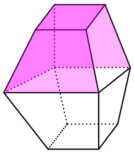

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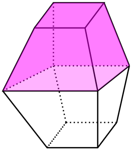

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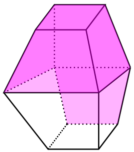

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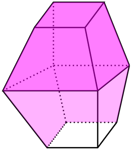

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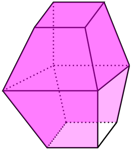

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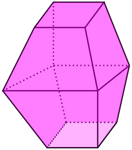

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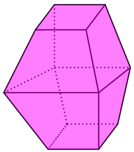

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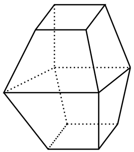
How can we use this to decompose our complex?

Relative complex– $\mathcal{C}/\mathcal{D}$ denotes a polytopal complex $\mathcal{C}$ with a subcomplex D removed.



We use our shelling order to define the following relative complexes for a subdivision $\mathcal{C}'$ of a shellable complex $\mathcal{C}$:

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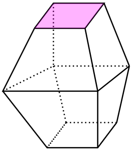

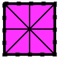

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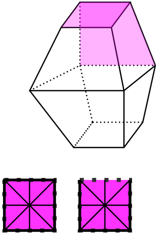

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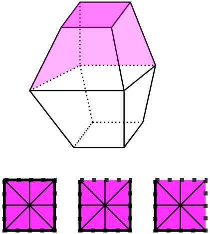

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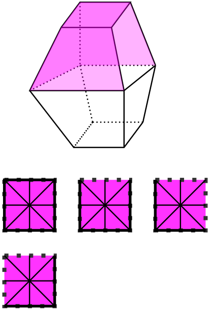

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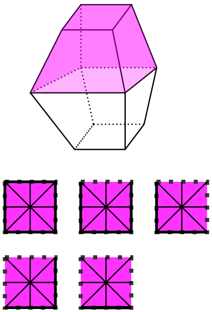

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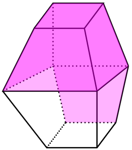

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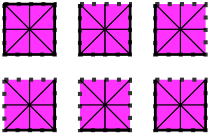

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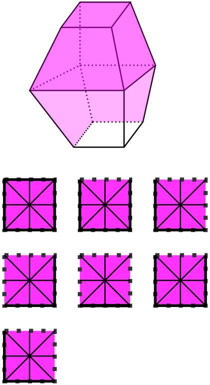

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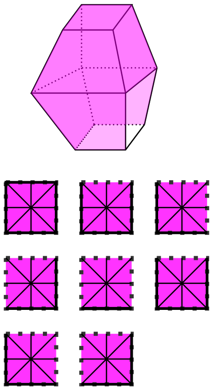

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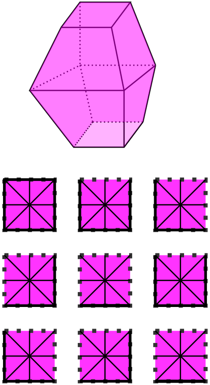

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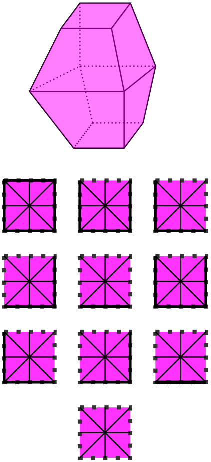

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We use our shelling order to define series of relative complexes.

Interlacing polynomials

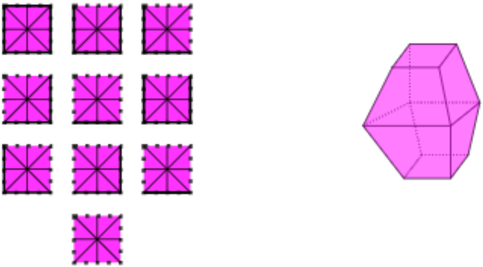
Interlacing polynomials: there is a zero of p between each pair of zeroes of q and vice versa for real-rooted polynomials p and q.

p $\prec$ q: p and q are interlacing and $p'q - q'p \geq 0$

Theorem (Borcea, Brändén) (2008) Suppose $p_1 \prec p_2 \prec \cdots \prec p_n$. Then any nonnegative combination of $p_1 \ldots p_n$ is real-rooted.

Putting it all together

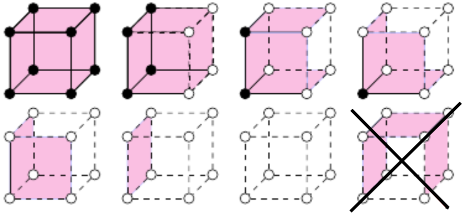
Theorem 1: H., Solus 2020
Let $\mathcal{C}$ be a shellable polytopal complex with shelling $(F_1, \ldots, F_s)$ and subdivision $\varphi : \mathcal{C}'$. If the h polynomials of the induced relative complexes form an interlacing sequence , then $h(\mathcal{C}'; x)$ is real-rooted.



Application to cubical complexes

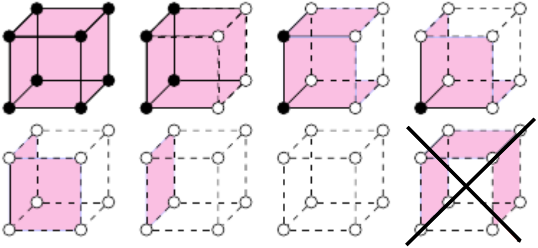
A shelling is **stable** if the relative complexes are stable (satisfying a specific property related to face lattices).

Cubical complexes: relative complexes look like:



Application to cubical complexes

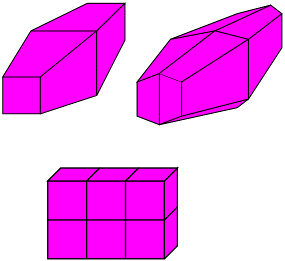
Theorem (H., Solus (2020)): The barycentric and rth edgewise subdivisions of cubical complexes admitting a stable shelling have real-rooted h-polynomials.



Examples of cubical complexes

The following cubical complexes have stable shellings.

- boundary complex of capped cubical polytopes
- boundary complex of cuboids
- piles of cubes



Further work

The overall technique outlined for barycentric subdivisions of certain cubical complexes also works for:

- barycentric subdivisions of simplicial products
- edgewise subdivisions of simplicial and certain cubical complexes.

Later, Athanasiadis proved that **all cubical polytopes** have a barycentric subdivision with a real-rooted h-polynomial.

Future work

- Are their other families of polytopes (not nec. cubical or simplicial) that admit stable shellings, and do real-rootedness results follow?
- Do all polytopes admit a **stable line shelling**?

Thank you all!!! :) :)